**ICS 2022 Problem Sheet #3**

**Problem 3.1: cartesian products**

Prove or disprove the following two propositions.

1. **(A ∩ B) × (C ∩ D) = (A × C) ∩ (B × D)**

(x,y) ∈ (A **∩** B) x (C **∩** D)

(x ∈ (A **∩** B)) (y ∈ (C **∩** D))

((x ∈ A) (x ∈ B)) ((y ∈ C) (y ∈ D))

((x ∈ A) (y ∈ C)) ((x ∈ B) (y ∈ D))

(x ∈ (A ×C)) (y ∈ (B ×D))

(x,y) ∈ (A × C) ∩ (C x D), which proves the propositions.

1. **(A ∪ B) × (C ∪ D) = (A × C) ∪ (B × D)**

(x,y) ∈ (A ∪ B) × (C ∪ D)

(x ∈ (A ∪ B)) (y ∈ (C ∪ D))

((x ∈ A) (x ∈ B)) ((y ∈ C) (y ∈ D))

(((x ∈ A) (x ∈ B)) (y ∈ C)) (((x ∈ A) (x ∈ B)) (y ∈ D))

(((x ∈ A) (y ∈ C)) ((x ∈ B) (y ∈ C))) (((x ∈ A) (y ∈ D)) ((x ∈ B) (y ∈ D)))

((x,y) ∈ (A × C) (x,y) ∈ (B × C)) ((x,y) ∈ (A × D) (x,y) ∈ (B × D))

(x,y) ∈ (A × C) ∪ (B × C) (x,y) ∈ (A × D) ∪ (B × D), which proves this is a wrong proposition

**Problem 3.2: reflexive, symmetric, transitive**

For each of the following relations, determine whether they are reflexive, symmetric, or transitive. Provide a reasoning.

1. The absolute difference of the integer numbers a and b is less than or equal to 3.

R = { (a, b) | a, b ∈ ∧ |a − b| ≤ 3 }

* Reflexive iff ∀a ∈ . (a, a) ∈ R

Let a ∈ (a, a) ∈ R |a − a| ≤ 3 that is equal to |0| ≤ 3, which is true ∀a ∈ .

R is reflexive ∀a ∈

* Symmetric iff ∀a, b ∈ . (a, b) ∈ R ⇒ (b, a) ∈ R

Let (a, b) ∈ R a, b ∈ and |a − b| ≤ 3

(b, a) ∈ R b, a ∈ , true as proved before, and |b − a| ≤ 3 |−(−b + a)| ≤ 3

|a − b| ≤ 3, true as proved before ⇒ (b, a) ∈ R

R is symmetric ∀a, b ∈

* Transitive iff ∀a, b, c ∈ . ((a, b) ∈ R ∧ (b, c) ∈ R) ⇒ (a, c) ∈ R

Let (a, b) ∈ R ∧ (b, c) ∈ R a, b, c ∈ and |a − b| ≤ 3 and |b − c| ≤ 3

Let (a, b) = (5, 2). 5, 2 ∈ and |5 − 2| ≤ 3 (5, 2) ∈ R

(b, c) = (2, 1). 2, 1 ∈ and |2 − 1| ≤ 3 (2, 1) ∈ R

But |5 − 1|3 so (5, 1) = (a, c) R R is not transitive ∀a, b, c ∈

1. The last digit of the decimal representation of the integer numbers a and b is the same.

R = { (a, b) | a, b ∈ ∧ (a mod 10) = (b mod 10) }

* Reflexive iff ∀a ∈ . (a, a) ∈ R

Let a ∈ (a, a) ∈ R (a mod 10) = (a mod 10), which is true ∀a ∈ .

R is reflexive ∀a ∈

* Symmetric iff ∀a, b ∈ . (a, b) ∈ R ⇒ (b, a) ∈ R

Let (a, b) ∈ R a, b ∈ and (a mod 10) = (b mod 10)

(b, a) ∈ R b, a ∈ , true as proved before, and (a mod 10) = (b mod 10), also true as proved before ⇒ (b, a) ∈ R

R is symmetric ∀a, b ∈

* Transitive iff ∀a, b, c ∈ . ((a, b) ∈ R ∧ (b, c) ∈ R) ⇒ (a, c) ∈ R

Let (a, b) ∈ R ∧ (b, c) ∈ R a, b, c ∈ and (a mod 10) = (b mod 10) ∧ (b mod 10) = (c mod 10) (a mod 10) = (c mod 10) ( a, c ∈ ) ⇒ (a, c) ∈ R

R is transitive ∀a, b, c ∈

**Problem 3.3: total, injective, surjective, bijective functions**

Are the following functions total, injective, surjective, or bijective? Explain why or why not.

1. f : → : with f(x) =

total: ∀x ∈ y ∈ with (x, y) ∈ f .

x ∈ there is exactly one f(x) = = y ∈ f is a total function

injective: ∀x, y ∈ . f (x) = f (y) ⇒ x = y

Let x, y ∈ and f (x) = f (y) ⇒ ⇒ ⇒ and x, y ∈

⇒ x = y ⇒ f is an injective function

surjective: ∀y ∈ . ∃x ∈ . f (x) = y

Let y ∈ and f (x) = y ⇒ ⇒ ⇒ which doesn’t always ∈

⇒ f is not an surjective function ⇒ f is also not a bijective function

1. f : → with f(x) =

total: ∀x ∈ y ∈ with (x, y) ∈ f .

x ∈ there is exactly one f(x) = = y ∈ f is a total function

injective: ∀x, y ∈ . f (x) = f (y) ⇒ x = y

Let x, y ∈ and f (x) = f (y) ⇒ ⇒ ⇒ and x, y ∈

⇒ x = - y or x = y ⇒ f is not an injective function

surjective: ∀y ∈ . ∃x ∈. f (x) = y

Let y ∈ and f (x) = y ⇒ ⇒ ⇒ ∈

⇒ f is not an surjective function ⇒ f is also not a bijective function

**Problem 3.4: types (Haskell)**

1. What is the type signature of the zip function? How many type variables appear in the type signature? Could it be more or less? Explain [The type signature of the zip function is: **zip :: [a] -> [b] -> [(a, b)]**](http://zvon.org/other/haskell/Outputprelude/RaSQ-NQRbSQ-NQRTa,bUS_t.html)

There are 2 type variables in the type signature: **Num ([a], [b])** and **(Num, Num) ([(a, b)])**

There can’t be more nor less type variables because the zip function itself has to have exactly 2 arguments, same type variable, and returns one different type variable, so it can only have exactly two type variables.

1. What are the types of the following expressions? Explain why!

2 + 3 **Num a => a because both 2 and 3 can be considered any number type**

2 + 9 `div` 3 **Integral a => a because the result of `div` is an integral**

2 + 9 / 3 **Rational (Fractional) a => a because the result of `/` is a rational number**

2 + sqrt 9 **Double a => a because the result of `sqrt` is a double/ floating point number**